MECHANICAL IMPEDANCE ANALYSIS FOR LUMPED PARAMETER MULTI-DEGREE OF FREEDOM/MULTI-DIMENSIONAL SYSTEMS

by Frank J. On

Goddard Space Flight Center
Greenbelt, Md.

National Aeronautics And Space Administration

ERRATUM

NASA Technical Note D-3865
May 1967

MECHANICAL IMPEDANCE ANALYSIS FOR LUMPED PARAMETER MULTI-DEGREE OF FREEDOM/MULTI-DIMENSIONAL SYSTEMS

by Frank J. On

Replace the first paragraph on Page 8 with the following:

. . . with the understanding that all quantities now refer to exterior co-ordinates. The impedance matrices $[Z^I]$ and $[Z^{II}]$ are roughly the counterparts of the "driving point" impedances, looking back from the interface commonly associated with one-dimensional interconnected systems.

MECHANICAL IMPEDANCE ANALYSIS

FOR LUMPED PARAMETER MULTI-DEGREE

OF FREEDOM/MULTI-DIMENSIONAL SYSTEMS

By Frank J. On

Goddard Space Flight Center
Greenbelt, Md.

ABSTRACT

Theoretical methods are presented for the analytic and experimental determination of mechanical impedance and its associated effects on the dynamic response of linear, multi-degree of freedom/multi-dimensional systems. Impedance matrices are formulated which are useful in describing and predicting the frequency dependent characteristic properties of these systems. The problem of interconnecting subsystems is formulated in terms of composite impedance matrices and interconnection equations, from which critical frequencies (resonances and anti-resonances) and dynamic response of the composite system may be obtained. Analytical and experimental procedures for the evaluation of these matrices are also discussed. To demonstrate the application of some of the procedures, an example problem is solved. Moreover, the results of this study are sufficiently general for extending the approach to many types of distributed system configurations subject to varied excitation phenomena.

CONTENTS

MECHANICAL IMPEDANCE ANALYSIS
FOR LUMPED PARAMETER MULTI-DEGREE
OF FREEDOM/MULTI-DIMENSIONAL SYSTEMS

by
Frank J. On
Goddard Space Flight Center

1. INTRODUCTION

The analysis of mechanical systems under dynamic loadings requires that the dynamic characteristics of the systems be known in addition to the characteristics of the excitations. In complex linear systems, it is useful to specify these system dynamic characteristics in terms of mechanical impedance (References 1 and 2).

In this study a mechanical impedance* analysis of complex systems is developed. This analysis ultimately may serve as an analytical tool in the computation of impedances and as a guideline in the laboratory measurement of impedances.

The basic philosophy adopted in this analysis is based on the idea of breaking up a complex system into component parts with simple dynamic properties of mass, stiffness, and damping that can be formulated readily in matrix notation. Appropriate component matrices are then considered as building blocks, that when fitted together according to a set of predetermined rules, provide the mechanical impedance characteristic of the entire system. Such a philosophy has been practiced in the well known method of influence coefficients (and related methods) of structural analysis. The matrix formulation of mechanical impedance may be considered as a natural extension of these methods, and like them it is well suited for processing by digital computers.

The results of these extensions to the determination of mechanical impedance require the re-interpretation of matrix parameters in terminologies that can be associated with the basic concepts of impedance. This is accomplished by manipulating equations of motions to appropriate forms physically corresponding to impedance quantities such as point and transfer impedances, free velocity and blocked force, etc.

---

*Although mechanical impedance and mobility are respectively defined as the complex ratio of sinusoidal force to sinusoidal velocity and of sinusoidal velocity to sinusoidal force, four other ratios between sinusoidal force and sinusoidal displacement and acceleration are equally useful. Thus the developments here may refer to all six ratios.

In order to present a method of analysis that is not confined to any particular type of structural system, a general method based on multi-degree of freedom, multi-dimensional systems* is adopted. The necessity of treating real complex structural systems as multi-dimensional systems has been recognized among environmental engineers for many years, but impedance concepts for multi-dimensional systems have been advocated only recently. This is largely due to the recognition that as aerospace structural systems become larger and more complex, one-dimensional impedance concepts are inadequate (Reference 3). Also, as the demands for larger environmental test facilities to accommodate these systems become more impractical, the use of subassembly and component testing philosophies becomes imperative. It is hoped that an appreciation of the generalized mechanical impedance approach will be gained from the developments herein. In particular, two aspects of the approach will be emphasized: (a) it provides both a means of analyzing complex structural systems by partitioning into successive smaller systems, and a natural means for matching boundary conditions; and (b) it provides a useful, sound basis for performing and correlating subassembly and component environmental tests, obviating the requirement for huge test facilities.

In the following sections, the theoretical relations and methods of evaluating the pertinent parameters required for the successful application of this concept are presented.

2. EQUATIONS OF MOTION

It is well known that complex linear time invariant mechanical systems can be analyzed in terms of their lumped parameter equivalents. The matrix equation of motion for such a grid network of lumped elements is given by:

$$[m_{\alpha\beta}]\{\ddot{x}_\beta(t)\} + [c_{\alpha\beta}]\{\dot{x}_\beta(t)\} + [k_{\alpha\beta}]\{x_\beta(t)\} = \{f_\alpha(t)\}, \tag{1}$$

where x denotes generalized coordinates and N denotes the number of degrees of freedom. The description of a particular system is contained in the coefficient matrices $[m_{\alpha\beta}]$, $[c_{\alpha\beta}]$ and $[k_{\alpha\beta}]$ [†]; the type of excitation is described by the column matrix $\{f_\alpha\}$.

In summation form, Equation 1 may be written as:

$$\sum_{\beta=1}^{N}\left(m_{\alpha\beta}\frac{d}{dt} + c_{\alpha\beta} + k_{\alpha\beta}\int dt\right)\dot{x}_\beta(t) = f_\alpha(t), \qquad \alpha = 1, 2, \cdots, N. \tag{2}$$

---

*Multi (or single) degree of freedom systems whose motions are confined along a single dimensional space are considered one-dimensional. Systems which are not one-dimensional are consequently both multi-dimensional and multi-degree of freedom.

[†]$c_{\alpha\beta}$ is the damping force at x_α due to a unit velocity at x_β; $k_{\alpha\beta}$ is the elastic force at x_α due to a unit displacement at x_β; $m_{\alpha\beta}$ contains scalar information regarding masses as well as first and second moments of the masses. In a simple system consisting of mass particles connected by springs and dash-pots, having no coupling, $m_{\alpha\alpha}$ would be either the mass or the principal moment of inertia of the αth particle. All $m_{\alpha\beta}$ transfer masses, first products and crossproducts of inertia would then be zero.

3. MECHANICAL IMPEDANCE APPROACH

Mechanical impedance is formally defined for linear systems over the domain of frequency as the ratio of the Fourier transforms of the force excitation and the velocity response. By Fourier transformation, Equation 2 may be written as the set

$$\sum_{\beta=1}^{N} \left(i\omega m_{\alpha\beta} + c_{\alpha\beta} + k_{\alpha\beta}/i\omega \right) V_{\beta}(\omega) = F_{\alpha}(\omega) , \qquad \alpha = 1, 2, \cdots, N , \tag{3}$$

where

$$\omega = 2\pi f = \text{circular frequency,}$$

$$V_{\beta}(\omega) = \text{Fourier transform of velocity, } \dot{x}_{\beta}(t),$$

$$F_{\alpha}(\omega) = \text{Fourier transform of force excitation, } f_{\alpha}(t) .$$

Letting

$$z_{\alpha\beta}(\omega) = i\omega m_{\alpha\beta} + c_{\alpha\beta} + k_{\alpha\beta}/i\omega \tag{4}$$

represent the frequency-dependent characteristic property of the system, Equation 3 may be written as

$$\sum_{\beta=1}^{N} Z_{\alpha\beta} V_{\beta} = F_{\alpha} , \tag{5}$$

or, in the shorthand matrix form,

$$\left[Z_{\alpha\beta} \right] \left\{ V_{\beta} \right\} = \left\{ F_{\alpha} \right\} . \tag{6}$$

In these equations, $Z_{\alpha\beta}$ are complex numbers giving the ratio of the transform vibratory force at coordinate α to the transform vibratory velocity at coordinate β. They are functions of frequency and may be called impedance parameters of the system. For $\alpha = \beta$, the $Z_{\alpha\beta}$ are called point impedance parameters, and for $\alpha \neq \beta$, they are called transfer impedance parameters. Alternatively, $Z_{\alpha\beta}$ may be considered as the ratio of vibratory force input at the α coordinate to vibratory velocity response at the β coordinate when all other coordinates are infinitely restrained (i.e., zero velocities). Accordingly, the matrix $\left[Z_{\alpha\beta} \right]$ may be considered as the mechanical impedance matrix of the system, while the column matrix $\left\{ V_{\beta} \right\}$ represents the transform of velocities corresponding to the transform of input forces $\left\{ F_{\alpha} \right\}$.

Heretofore, the developments have been analogous to those of the methods of dynamic influence and stiffness coefficients. Equation 6, however, expressed in terms of all the coordinates which have been defined, may not be in the most useful form. In many practical situations, some of these coordinates are not of interest either because impedance information at these coordinates is not desired, or these coordinates are not accessible for physical measurement. Frequently, the coordinates of interest are those at which subsequent interconnection with another system is anticipated. Consequently, it is desirable or necessary to express Equation 6 in terms of matrix quantities referred to only the coordinates of interest, hereafter called "exterior coordinates." The coordinates not of interest are called "interior coordinates."

If Equation 6 is rearranged, grouped and partitioned according to "interior" and "exterior" applied forces such that n coordinates are exterior and the remaining p coordinates are interior, Equation 6 may be written as

$$\begin{bmatrix} \begin{bmatrix} Z_{EE} \end{bmatrix} & \begin{bmatrix} Z_{EI} \end{bmatrix} \\ {}_{nn} & {}_{np} \\ \begin{bmatrix} Z_{IE} \end{bmatrix} & \begin{bmatrix} Z_{II} \end{bmatrix} \\ {}_{pn} & {}_{pp} \end{bmatrix} \begin{Bmatrix} \{V_E\} \\ {}_{n1} \\ \{V_I\} \\ {}_{p1} \end{Bmatrix} = \begin{Bmatrix} \{F_E\} \\ {}_{n1} \\ \{F_I\} \\ {}_{p1} \end{Bmatrix} . \tag{7}$$

The subscripted $[Z]$, $\{V\}$ and $\{F\}$ are the submatrices corresponding to the partition; the subscripts E and I denote respectively exterior and interior. In accordance with the reciprocity theorem for linear systems,

$$\begin{bmatrix} Z_{EI} \end{bmatrix} = \begin{bmatrix} Z_{IE} \end{bmatrix}^T , \tag{8}$$

where T denotes transpose.

The expanded form of Equation 7 provides a set of equations from which we can develop the matrices entirely in terms of quantities referred to exterior coordinates. Obtaining the submatrix $\{V_I\}$ from the second equation of the partitioned form and substituting in the first equation yields

$$\begin{bmatrix} Z_E \end{bmatrix} \{V_E\} = \{F_E - F_E^b\} , \\ {}_{nn} \quad {}_{n1} \qquad {}_{n1} \tag{9}$$

where

$$\begin{bmatrix} Z_E \end{bmatrix} = \begin{bmatrix} Z_{EE} \end{bmatrix} - \begin{bmatrix} Z_{EI} \end{bmatrix} \begin{bmatrix} Z_{II} \end{bmatrix}^{-1} \begin{bmatrix} Z_{IE} \end{bmatrix} , \\ {}_{nn} \qquad {}_{nn} \qquad {}_{np} \quad {}_{pp} \quad {}_{pn} \tag{10}$$

$$\{F_E^b\} = \begin{bmatrix} Z_{EI} \end{bmatrix} \begin{bmatrix} Z_{II} \end{bmatrix}^{-1} \{F_I\} , \\ {}_{n1} \qquad {}_{np} \quad {}_{pp} \quad {}_{p1} \tag{11}$$

and $\begin{bmatrix} Z_{II} \end{bmatrix}^{-1}$ is the inverse matrix of $\begin{bmatrix} Z_{II} \end{bmatrix}$, which is assumed nonsingular.

Equation 9 may be termed the "exterior equation" of the system since all quantities now refer to exterior coordinates. The matrix $[Z_E]$ in this equation then represents the internal impedance as seen from the exterior points, and differs conceptually from $[Z_{\alpha\beta}]$ in the original matrix equation only in the coordinates to which it is referred. Here the matrix $\{F^b\}$ is defined as the "blocked force matrix" of the system, and it represents the interior excitations as transmitted to the exterior coordinates for exterior coordinates fixed so that no motion occurs. In Equation 9, the negative sign with F_E^b denotes the change of reactions due to interior forces into equivalent externally applied forces.

If a system contains no internal source of energy, the matrix of interior forces, $\{F_I\}$, and consequently the matrix of blocked forces, $\{F_E^b\}$, becomes identically zero. Such a system is termed a "passive" system as opposed to an "active" system when $\{F_E^b\}$ is nonzero. According to the foregoing developments, the activeness or passiveness of any system may be arbitrary in that it depends primarily on the initial definition of interior and exterior coordinates.

It follows from Equation 9 that the matrix of exterior velocities $\{V_E\}$ can be obtained explicitly by premultiplying through by the inverse matrix of $[Z_E]$ which is assumed nonsingular,

$$\{V_E\} \;=\; [Z_E]^{-1}\{F_E - F_E^b\} \; . \tag{12}$$

The inverse matrix $[Z]^{-1}$ is termed the "matrix of mobility" $[Y]$. Frequently the use of $[Y]$ is more desirable than $[Z]$ (see Section 7.3). At this point the choice is trivial. When the matrix $\{F_E\}$ of applied forces at exterior coordinates is zero, the velocity matrix of Equation 12 is significant enough to warrant the special term "free velocity" matrix, expressed by

$$\{V_{E0}\} \;=\; - [Z_E]^{-1}\{F_E^b\} \; . \tag{13}$$

In terms of the free velocity matrix, Equation 12 becomes

$$\{V_E - V_{E0}\} \;=\; [Z_E]^{-1}\{F_E\} \; . \tag{13'}$$

An alternative form of Equation 9 may be obtained in terms of the free velocity matrix and the mobility matrix (inverse matrix of impedance) of the system, as follows. Premultiplying Equation 6 by the inverse matrix $[Z_{\alpha\beta}]^{-1}$, which will be denoted by $[Y_{\alpha\beta}]$ yields

$$\{V_\alpha\} \;=\; [Y_{\alpha\beta}]\{F_\beta\} \; , \tag{14}$$

where $[Y]$ is the mobility matrix.

Partitioning in terms of exterior and interior coordinates, Equation 14 can be written as

$$\left\{ \begin{array}{c} \{V_E\} \\ {\scriptstyle n1} \\ \{V_I\} \\ {\scriptstyle p1} \end{array} \right\} = \begin{bmatrix} [Y_{EE}] & [Y_{EI}] \\ {\scriptstyle nn} & {\scriptstyle np} \\ [Y_{IE}] & [Y_{II}] \\ {\scriptstyle pn} & {\scriptstyle pp} \end{bmatrix} \left\{ \begin{array}{c} \{F_E\} \\ {\scriptstyle n1} \\ \{F_I\} \\ {\scriptstyle p1} \end{array} \right\} \; . \tag{15}$$

5

Expanding Equation 15 provides a set of equations from which we can develop the matrices entirely in terms of quantities referred to exterior coordinates. Obtaining the submatrix $\{F_I\}$ from the second equation of the partitioned form and substituting in the first equation yields:

$$\underset{n1}{\{V_E - V_{E0}\}} = \underset{nn}{[Y_E]} \underset{n1}{\{F_E\}} ,$$

(16)

where

$$\underset{nn}{[Y_E]} = \underset{nn}{[Y_{EE}]} - \underset{np}{[Y_{EI}]} \underset{pp}{[Y_{II}]^{-1}} \underset{pn}{[Y_{IE}]} ,$$

(17)

$$\underset{n1}{\{V_{E0}\}} = \underset{np}{[Y_{EI}]} \underset{pp}{[Y_{II}]^{-1}} \underset{p1}{\{V_I\}} .$$

(18)

The determinant of $[Z_E]$ in Equation 9, or of $[Y_E]$ in Equation 16 (when damping is neglected) may be recognized as the frequency determinant of the system observed from the exterior coordinates. Upon expansion it yields a polynomial in ω^2, the roots ω_n of which are known as the natural frequencies. Cooresponding to each value of ω_n, an amplitude matrix may be computed for each natural mode.

4. MULTI-DIMENSIONAL IMPEDANCE

The multi-dimensional or one-dimensional aspect of a system is characterized by the nature of the matrices of Equation 7. The necessary and sufficient condition for one-dimensional systems is uniformity in coordinate types (rectilinear or rotational) which define excitation and response of the system confined to a single dimensional space. For example, a single terminal having three rectilineal and/or rotational types of coordinates is not one-dimensional within the scope of this definition.

The general nature of multi-dimensional systems invariably requires consideration of both rectilinear and rotational types of impedance analysis and measurement. Rearranging and grouping with respect to rectilinear and rotational impedance, as well as to interior and exterior coordinates, and partitioning accordingly, Equation 6 may be expressed as

$$
\begin{bmatrix}
\underset{n_1 n_1}{[Z_{EE}^{\delta\delta}]} & \underset{n_1 n_2}{[Z_{EE}^{\delta\theta}]} & \underset{n_1 p_1}{[Z_{EI}^{\delta\delta}]} & \underset{n_1 p_2}{[Z_{EI}^{\delta\theta}]} \\
\underset{n_2 n_1}{[Z_{EE}^{\theta\delta}]} & \underset{n_2 n_2}{[Z_{EE}^{\theta\theta}]} & \underset{n_2 p_1}{[Z_{EI}^{\theta\delta}]} & \underset{n_2 p_2}{[Z_{EI}^{\theta\theta}]} \\
\underset{p_1 n_1}{[Z_{IE}^{\delta\delta}]} & \underset{p_1 n_2}{[Z_{IE}^{\delta\theta}]} & \underset{p_1 p_1}{[Z_{II}^{\delta\delta}]} & \underset{p_1 p_2}{[Z_{II}^{\delta\theta}]} \\
\underset{p_2 n_1}{[Z_{IE}^{\theta\delta}]} & \underset{p_2 n_2}{[Z_{IE}^{\theta\theta}]} & \underset{p_2 p_1}{[Z_{II}^{\theta\delta}]} & \underset{p_2 p_2}{[Z_{II}^{\theta\theta}]}
\end{bmatrix}
\begin{Bmatrix}
\underset{n_1 1}{\{V_E^{\delta}\}} \\
\underset{n_2 1}{\{V_E^{\theta}\}} \\
\underset{p_1 1}{\{V_I^{\delta}\}} \\
\underset{p_2 1}{\{V_I^{\theta}\}}
\end{Bmatrix}
=
\begin{Bmatrix}
\underset{n_1 1}{\{F_E^{\delta}\}} \\
\underset{n_2 1}{\{F_E^{\theta}\}} \\
\underset{p_1 1}{\{F_I^{\delta}\}} \\
\underset{p_2 1}{\{F_I^{\theta}\}}
\end{Bmatrix} ,
$$

(19)

where the superscripts δ and θ respectively represent rectilinear and rotational. It is apparent in Equation 19 that there are four possible types of impedance:

$[z^{\delta\delta}]$ = point or transfer impedances defined by ratios of rectilinear forces to rectilinear velocities,

$[z^{\delta\theta}]$ = transfer impedances defined by ratios of rectilinear forces to rotational velocities,

$[z^{\theta\delta}]$ = transfer impedances defined by ratios of couples to rectilinear velocities,

$[z^{\theta\theta}]$ = point or transfer impedances defined by ratios of couples to rotational velocities.

In accordance with the reciprocity theorem for linear systems, these impedances and their matrices possess the following convenient transpose properties:

$$[Z_{EE}^{\delta\theta}] = [Z_{EE}^{\theta\delta}]^T, \qquad [Z_{II}^{\delta\theta}] = [Z_{II}^{\theta\delta}]^T$$

$$[Z_{EI}^{\delta\delta}] = [Z_{IE}^{\delta\delta}]^T, \qquad [Z_{EI}^{\theta\theta}] = [Z_{IE}^{\theta\theta}]^T$$

$$[Z_{EI}^{\delta\theta}] = [Z_{IE}^{\theta\delta}]^T, \qquad [Z_{EI}^{\theta\delta}] = [Z_{IE}^{\delta\theta}]^T. \qquad (20)$$

5. INTERCONNECTION OF SYSTEMS

In Section 3 it was shown how the performance equation of a system can be expressed in terms of coordinates of anticipated interconnections, heretofore defined as exterior coordinates. Let the set of these exterior coordinates corresponding to $\{V_E\}$ be common to two systems which are otherwise separated. Block diagrams representing the two systems are shown in Figure 1a. The systems will be connected as shown by Figure 1b, and their combined behavior will be examined in terms of interfacial critical (resonant and anti-resonant) frequencies and response.

The individual performance of the systems prior to connection can be described with reference to Equation 9 by the set of exterior equations:

$$\underset{nn}{[z^I]} \underset{n1}{\{V^I\}} = \underset{n1}{\{F^I - F^{b\,I}\}}, \qquad (21)$$

$$\underset{mm}{[z^{II}]} \underset{m1}{\{V^{II}\}} = \underset{m1}{\{F^{II} - F^{b\,II}\}} \qquad (22)$$

where the superscripts I and II designate a specific system. The subscript E is dropped

a. Systems before interconnection.
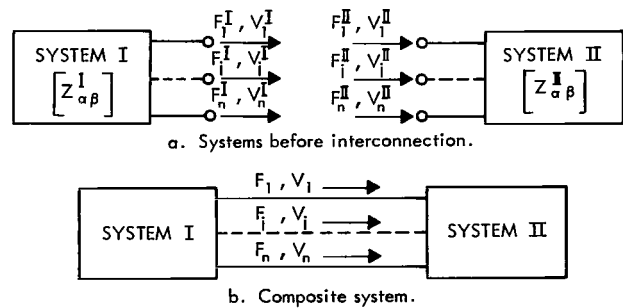
b. Composite system.

Figure 1—Block diagrams of (a) subsystems and (b) composite system.

with the understanding that all quantities now refer to exterior coordinates. The impedance matrices $[Z^I]$ and $[Z^{II}]$ are roughly the counterparts of the "driving point" impedances, looking back from the interface commonly associated with one-dimensional interconnected systems.

Assuming the identification of coordinates of system I to be physically and mathematically compatible with those of system II, then $m = n$ in Equations 21 and 22. When the systems are combined to form a composite system, Equations 21 and 22 must satisfy a force equilibrium condition

$$\{R\} = \left\{ F^I + F^{II} - \left(F^{b^I} + F^{b^{II}}\right) \right\} , \tag{23}$$

and a velocity compatibility condition*

$$\{V\}_{n1} = \left\{V^I\right\}_{n1} = \left\{V^{II}\right\}_{n1} , \tag{24}$$

where the nonsuperscripted quantities refer to composite quantities at the interface.

Satisfying these conditions in Equations 21 and 22, and subsequently summing up these equations yields

$$[H]_{nn} \{V\}_{n1} = \{R\}_{n1} , \tag{25}$$

where

$$[H]_{nn} = \left[Z^I + Z^{II}\right]_{nn} . \tag{26}$$

Equation 25 is termed the "composite systems equation," which is the equation of equilibrium in terms of the interconnection velocity $\{V\}$ for the complete system and the matrix $[H]$, called the "interconnection matrix."

Thus, in the calculation of the interconnection velocity, the complete system has been regarded as an assembly of the subsystems subjected to the equivalent loading $\{R\}$ of Equation 23. The summation $\left\{F^{b^I} + F^{b^{II}}\right\}$ implies addition of the corresponding interconnection reactions for interconnections blocked (i.e., blocked forces), while $\left\{F^I + F^{II}\right\}$ is the loading matrix for externally applied forces on these interconnections.

In the case of free vibrations,

$$\left\{F^I\right\} = \left\{F^{II}\right\} = \left\{F^{b^I}\right\} = \left\{F^{b^{II}}\right\} = 0 , \tag{27}$$

---

*The nature of the mechanical connections at the interconnection (i.e., boundary conditions) are implicitly defined by the force equilibrium and velocity compatibility equations.

and consequently

$$\{R\} = \{0\} , \tag{28}$$

and Equation 25 reduces to

$$[H] \{V\} = \{0\} , \tag{29}$$

a set of linear homogeneous equations in V. The roots of the frequency determinant

$$\Delta(H) = 0 \tag{30}$$

correspond to the natural frequencies of the composite system observed at the interconnection.

6. ANALYTICAL APPROACH

In order to evaluate the matrices of the foregoing sections, analytical and experimental procedures are recommended in the sections that follow. The application of these procedures in conjunction with automatic data processing techniques will greatly enhance the computational schemes required by theory.

The basic procedure for analytical determination of the desired matrices discussed in the foregoing sections may be summarized as follows:

(i) Choose a set of appropriate grid points for the system and identify the required generalized coordinates. In terms of these coordinates, obtain the matrix equation of motions from which the coefficient matrices $[m_{\alpha\beta}]$, $[k_{\alpha\beta}]$ and $[c_{\alpha\beta}]$ are determined.

(ii) Determine the matrix of applied forces (or moments) $\{F\}$.

(iii) By elementary matrix operations, determine the submatrices $[Z_{EE}]$, $[Z_{EI}]$, $[Z_{IE}]$, $[Z_{II}]$, $[Z_{II}]^{-1}$, $\{V_E\}$, $\{V_I\}$, $\{F_E\}$ and $\{F_I\}$ from $[Z_{\alpha\beta}]$, $\{F_\alpha\}$ and $\{V_\beta\}$.

(iv) Determine $[Z_E]$ and $\{F_E^b\}$ from Equations 10 and 11.

(v) Determine the response $\{V_E\}$ from Equation 12.

(vi) Setting the damping coefficients to zero in the frequency determinant, determine the critical (resonant and anti-resonant) frequencies and the corresponding natural modes.

If the effect of connecting a second system to the first system is desired, the following additional steps are required:

(vii) Repeat steps (i)—(vi) for the second system.

(viii) Determine the interconnection matrix [H] from Equation 26.

9

(ix) Determine the loading matrix {R} from Equation 23.

(x) Determine the response at the interconnection from Equation 25.

(xi) Neglecting damping, determine critical (resonant and anti-resonant) frequencies of composite system from Equation 30 and compute corresponding natural modes.

7. MEASUREMENT APPROACH

Experimental measurement of the impedance matrix $[Z_E]$ requires the acquisition of a combination of experimental data which will supply sufficient information for evaluating the matrix elements of $[Z_E]$. The following schemes for experimentally determining matrix $[Z_E]$ are based on the fact that if a performance equation (Equation 9 or 16) is to truly describe the dynamic behavior of the system, matrix $[Z_E]$ must be compatible with the set of velocities $\{V_E\}$ and forces $\{F_E\}$ measured on the system at the specified coordinates.

7.1. Determination of Impedance Parameters

7.1.1 Active Systems

In the experimental determination of impedance parameters, the measurement of free velocity V_0 is more readily accomplished than that of blocked force F^b. This is due to the fact that ideal physical restraint conditions required for block force measurement are seldom realizable. By substituting Equation 13 into Equation 12, the performance equation for active systems may be written in modified form as

$$[Z] \{V - V_0\} = \{F\} , \tag{31}$$

where the subscript E has been dropped. Hereafter, the absence of E implies that all coordinates are exterior. For a system of n coordinates, the number of impedance parameters will be n^2. Theoretically, only $n(n+1)/2$ number will have different values, since $Z_{\alpha\beta} = Z_{\beta\alpha}$, for $(\alpha \neq \beta)$. It is highly desirable, however, to evaluate all n^2 parameters since this will provide a means of validating the assumption of linearity of the system by use of the symmetric condition $Z_{\alpha\beta} = Z_{\beta\alpha}$ $(\alpha \neq \beta)$. Since the column matrix of free velocities must also be determined, a total of $n+1$ tests are required at each frequency to evaluate all the impedance matrix elements and the elements of free velocity of Equation 31.

Test 1: With the system suspended on its natural restraints, or, in the case of a free system, on a low frequency suspension such that all exterior coordinates are unattached (zero load impedance), operate the system and determine the phasor* of all resulting free velocities. Denote these free velocities by $\{V_0^{(1)} (\omega)\}$ where the superscript (1) represents test number.

---

*The term "phasor," as used here, implies magnitude and phase of a steady state sinusoidal quantity where phase measurement is made with respect to an arbitrary fixed reference. If a quantity is not steady state sinusoidal, Fourier transform of the quantity is required for determining the phasors.

10

Test n + 1: With the system suspended as in Test 1, operate the system and apply an external force at one of the exterior coordinates (e.g., the ath coordinate) and determine the phasor of all resulting velocities at all exterior coordinates and the phasor of the applied force. Denote these velocity phasors and this applied force phasor respectively by $\{V^{(2)}(\omega)\}$ and $\{F^{(2)}(\omega)\}$, where all elements of $\{F^{(2)}\}$ except the ath element are zeros. Repeating this test n times, each time applying a force at successively different coordinates and determining the phasor of all resulting velocities and the applied force, provides a set of experimental data from which the following resulting matrix equations may be solved simultaneously for the elements of [Z] at any specified frequency, say ω_0:

$$\left[Z\left(\omega_0\right)\right]\left\{V^{(k)}\left(\omega_0\right) - V_0^{(1)}\left(\omega_0\right)\right\} = \left\{F^{(k)}\left(\omega_0\right)\right\} , \qquad (32)$$

where the test number k = 2, 3, $\cdots$, n + 1.

7.1.2 Passive Systems

For passive systems, the matrix of free velocities $\{V_0\}$ is identically zero and Equation 31 reduces to

$$[Z]\{V\} = \{F\} . \qquad (33)$$

Accordingly, tests of the type outlined under Test (n + 1) for active systems (except now there are no free velocities) provide a set of experimental data from which the resulting matrix equations (similar to Equation 32) which follow may be solved simultaneously for the elements of [Z] of the passive system at any specified frequency ω_0:

$$\left[Z\left(\omega_0\right)\right]\left\{V^{(k)}\left(\omega_0\right)\right\} = \left\{F^{(k)}\left(\omega_0\right)\right\} , \qquad (34)$$

where k = 1, 2, $\cdots$, n. Since passive and active systems are conceptually related by the absence or presence of an appropriately determined free velocity or blocked force matrix, these determinations based on Equation 34, if supplemented by an independent determination of free velocities, may also be employed for active systems.

7.2. Determination of Mobility Parameters

The solutions of Equation 32 or 34 when n > 3 will invariably require the use of digital computers. A more practical procedure which considers the computation of elements of the inverse of the impedance matrix (mobility matrix) can be made available as follows: Equation 32 in its inverse form,

$$\left\{V^k\left(\omega_0\right) - V_0^{(1)}\left(\omega_0\right)\right\} = \left[Z\left(\omega_0\right)\right]^{-1}\left\{F^{(k)}\left(\omega_0\right)\right\} ,$$

$$= \left[Y\left(\omega_0\right)\right]\left\{F^{(k)}\left(\omega_0\right)\right\} , \qquad (35)$$

where [Y] is the mobility matrix, and expanded, can be written as follows (assuming force excitation at a single coordinate to be taken in numerical order with the subscript of the coordinates):

$$V_1^{(2)}\left(\omega_0\right) - V_{0,1}^{(1)}\left(\omega_0\right) = Y_{11}\left(\omega_0\right) F_1^{(2)}\left(\omega_0\right) + 0 + 0 + \cdots + 0$$

$$V_2^{(2)}\left(\omega_0\right) - V_{0,2}^{(1)}\left(\omega_0\right) = Y_{21}\left(\omega_0\right) F_1^{(2)}\left(\omega_0\right) + 0 + 0 + \cdots + 0$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$V_n^{(2)}\left(\omega_0\right) - V_{0,n}^{(1)}\left(\omega_0\right) = Y_{n1}\left(\omega_0\right) F_1^{(2)}\left(\omega_0\right) + 0 + 0 + \cdots + 0$$

$$V_1^{(n+1)}\left(\omega_0\right) - V_{0,1}^{(1)}\left(\omega_0\right) = 0 + 0 + \cdots + Y_{1n}\left(\omega_0\right) F_n^{(n+1)}\left(\omega_0\right)$$

$$V_2^{(n+1)}\left(\omega_0\right) - V_{0,2}^{(1)}\left(\omega_0\right) = 0 + 0 + \cdots + Y_{2n}\left(\omega_0\right) F_n^{(n+1)}\left(\omega_0\right)$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$V_n^{(n+1)}\left(\omega_0\right) - V_{0,n}^{(1)}\left(\omega_0\right) = 0 + 0 + \cdots + Y_{nn}\left(\omega_0\right) F_n^{(n+1)}\left(\omega_0\right) \quad . \tag{36}$$

From Equations 36 the elements of the mobility matrix are readily computed as

$$Y_{\alpha\beta}\left(\omega_0\right) = \frac{V_\alpha^{(\beta+1)}\left(\omega_0\right) - V_{0,\alpha}^{(1)}\left(\omega_0\right)}{F_\beta^{(\beta+1)}\left(\omega_0\right)} , \tag{37}$$

where $\alpha, \beta = 1, 2, 3, \cdots, n$.

For the case of passive systems wherein no free velocities are present, Equation 36 and consequently Equation 37 reduces to

$$Y_{\alpha\beta}\left(\omega_0\right) = \frac{V_\alpha^{(\beta)}\left(\omega_0\right)}{F_\beta^{(\beta)}\left(\omega_0\right)} , \tag{38}$$

where the superscripts $(\beta + 1)$ have been modified to be compatible with the test number shown in Equation 34.

The complete determination of the impedance or mobility matrix requires performing the above procedures over all frequencies of interest. The impedance matrix [Z] may be obtained by performing the inverse of the mobility matrix, i.e., $[Z] = [Y]^{-1}$.

7.3. Significance of Mobility and Impedance Parameters

It is evident in the foregoing analytical procedures that the significance of mobility parameters versus impedance parameters is primarily one of utility. Since they are mathematically related

by the matrix inverse relation

$$[Z] = [Y]^{-1} ,$$
(39)

the choice may depend on whether force or velocity response is considered as the unknown, or may depend on which is the more desirable parameter for describing one's interest. In problems dealing with creating motions, the use of mobility may be preferred; in problems dealing with impeding motions, the use of impedance may be preferred.

A greater significance which is related to the measurement of these parameters is demonstrated by the invariant and variant property of the elements of the mobility and impedance matrix, respectively (Reference 4). Consider a system of three degrees of freedom as shown in Figure 2a. The performance equation of this system in terms of the mobility matrix is

$$\left\{ \begin{matrix} V_1 \\ V_2 \\ V_3 \end{matrix} \right\} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \left\{ \begin{matrix} F_1 \\ F_2 \\ F_3 \end{matrix} \right\} ,$$
(40)

where {V} is the column vector of resultant velocities corresponding to the column vector of applied forces {F}. The elements of the mobility matrix [Y] may be evaluated by the method of Section 7.2. When only F_1 is applied and x_1 is considered the exterior coordinate (Figure 2b), the performance equation is simply

a. Block representation of a one-dimensional/three degrees of freedom system.

b. Block representation when only F_1 and x_1 are considered.

c. Block representation when F_1, F_2, x_1, and x_2 are considered.
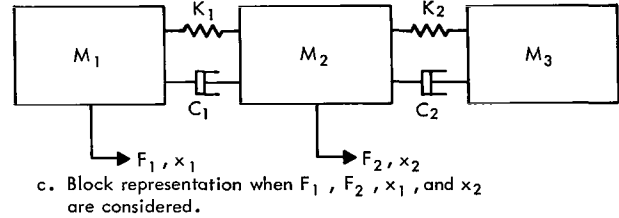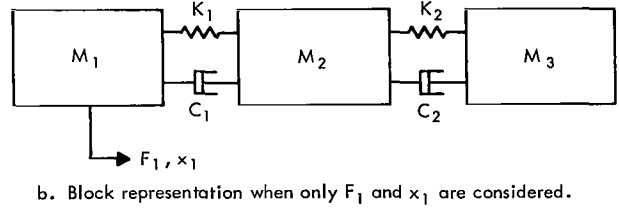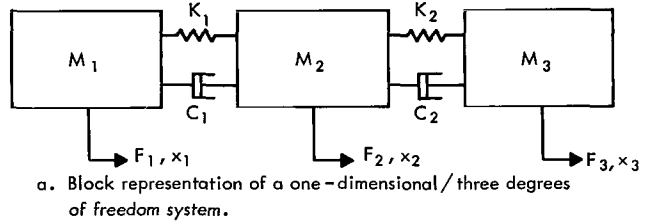
Figure 2—Block representation of a system with varying number of coordinates under consideration.

$$V_1 = Y_{11} F_1 ,$$

from which

$$Y_{11} = \frac{V_1}{F_1} .$$

The performance equation when only F_1 and F_2 are applied, and x_1 and x_2 are considered the exterior coordinates, (Figure 2c) is

$$\left\{ \begin{matrix} V_1 \\ V_2 \end{matrix} \right\} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \left\{ \begin{matrix} F_1 \\ F_2 \end{matrix} \right\} ,$$

13

from which the elements of [Y], evaluated by method of Section 7.2, are:

$$Y_{11} = \frac{V_1}{F_1}, \qquad Y_{12} = \frac{V_1}{F_2}, \qquad Y_{21} = \frac{V_2}{F_1}, \qquad Y_{22} = \frac{V_2}{F_2}.$$

It is noted that Y_{11} remains the same as when only x_1 is considered. Considering all three coordinates x_1, x_2 and x_3 with applied forces F_1, F_2 and F_3, the performance equation becomes

$$\begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix},$$

where the elements $Y_{\alpha\beta}$ (α, β = 1, 2) remain the same as when only x_1 and x_2 are considered and the additional elements are evaluated as

$$Y_{13} = \frac{V_1}{F_3}, \qquad Y_{23} = \frac{V_2}{F_3}, \qquad Y_{31} = \frac{V_3}{F_1}, \qquad Y_{32} = \frac{V_3}{F_2}, \qquad Y_{33} = \frac{V_3}{F_3}.$$

Accordingly, the elements of the mobility matrix for the system are seen to be invariant, in this manner, with respect to the number of coordinates considered. Now consider the computation of the elements of the impedance matrix using Equation 39. A general element may be computed from

$$Z_{\alpha\beta} = \frac{(-1)^{\alpha+\beta} A_{\alpha\beta}}{\Delta(Y)}, \qquad (41)$$

where the numerator and denominator are respectively the adjoint and determinant of the mobility matrix. As the numerator and denominator in Equation 41 are dependent on the order of the matrix [Y] and are in general not equal, the elements of the impedance matrix consequently are variant with respect to the number of coordinates considered. This implies that as additional coordinates are considered, each previous matrix element must be recomputed.

The invariant property of the mobility matrix makes it desirable and natural to measure and use mobility parameters, and to consider impedance parameters as derived quantities with certain utilities. Furthermore the use of the term "driving point," frequently associated with one-dimensional systems, possesses true counterparts in multi-dimensional systems when used only with elements of the mobility matrix.

For active systems, the choice of free velocity or blocked force measurements requires some consideration. Mathematically, the free velocity matrix is simply related to the blocked force matrix through Equation 13. Computationally, free velocities can be obtained from blocked forces if the mobility matrix is known. Experimentally however, the measurement of blocked forces in

14

general is exceedingly difficult because of the physical constraint required by theory. The measurement of free velocities is more easily accomplished and consequently the more natural quantities to measure.

8. DETERMINATION OF IMPEDANCE IN PRESENCE
OF PARTIAL EXTERNAL RESTRAINTS

In the laboratory, the impedance matrix $[Z_E]$ of a system sometimes cannot be determined readily for ideal free conditions (when the system is naturally free) or for natural restraints* (when the system is naturally restrained). Some of the free coordinates may be restrained to some degree, or some of the natural restraints may be altered. In some cases the required measurement instrumentation may impose significant restraints. A formal method to determine the impedance $[Z_E]$ of the free system from an impedance $[\overline{Z}_E]$ determined with the system unnaturally restrained in a known fashion is developed in the following.

Consider an active system having a class of u exterior coordinates which are free of restraints and another class of v exterior coordinates which are restrained in a known fashion. When the set of free coordinates is denoted by subscript f and the set of restrained coordinates by r, the system when entirely free of restraints (Figure 3a) may be expressed as

$$
\left\{
\begin{array}{c}
\{V_f\} \\
\text{\scriptsize u1} \\
\{V_r\} \\
\text{\scriptsize v1}
\end{array}
\right\}
=
\left[
\begin{array}{cc}
[Y_{ff}] & [Y_{fr}] \\
\text{\scriptsize uu} & \text{\scriptsize uv} \\
[Y_{rf}] & [Y_{rr}] \\
\text{\scriptsize vu} & \text{\scriptsize vv}
\end{array}
\right]
\left\{
\begin{array}{c}
\{F_f - F_f^b\} \\
\text{\scriptsize u1} \\
\{F_r - F_r^b\} \\
\text{\scriptsize v1}
\end{array}
\right\} ,
\qquad (42)
$$

which is obtained by substituting

$$[Z]^{-1} = [Y]$$

in Equation 12, and partitioning in terms of submatrices corresponding to free and restrained coordinates. When the restraints are expressed in terms of a known impedance matrix $[Z_e]$ as shown in Figure 3b, the application of the composite equation (Equation 25) at the restrained coordinates yields

$$[Z_{rr} + Z_e]\{\overline{V}_r\} = \{R\} , \qquad (43)$$
$$\text{\scriptsize vv} \quad\quad \text{\scriptsize v1} \quad\quad \text{\scriptsize v1}$$
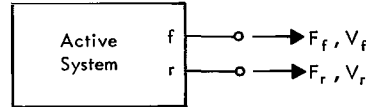
a. Block representation of "Free" (naturally restrained) active system.
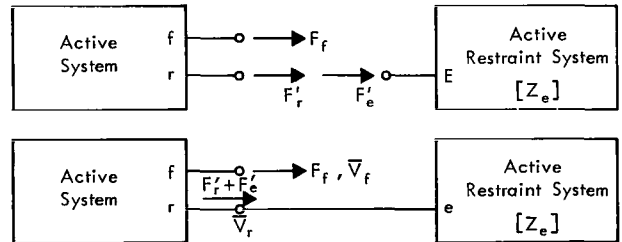
b. Block representation of restrained system.
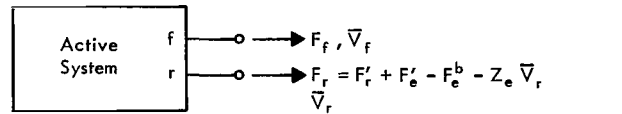
c. Equivalent block representation of Figure 3b.

Figure 3—Block representation of a system under (a) natural restraints, and (b, c) unnatural restraints.

*Hereafter, no distinction will be made between free and naturally restrained systems.

where

$$\{R\} = \{F_r' + F_e' - F_r^{\ b} - F_e^{\ b}\} ,$$ (44)

and $\{\bar{V}_r\}$ are the interconnection velocities. As shown in Figure 3b, $\{F_r' + F_e'\}$ are the externally applied forces at the restraints.

The experimental evaluation of [Y] of the restrained system which will be denoted by [Ȳ] may be accomplished in a manner similar to that of Section 7.2 by applying components of $\{F_f\}$ and $\{F_r' + F_e'\}$ one component at a time, measuring the resulting velocity at all coordinates, and finally solving for the elements of [Ȳ] in

$$
\left\{
\begin{array}{c}
\{\bar{V}_f\} \\
{}_{u1} \\
\{\bar{V}_r\} \\
{}_{v1}
\end{array}
\right\}
=
\left[
\begin{array}{cc}
[\bar{Y}_{ff}] & [\bar{Y}_{fr}] \\
{}_{uu} & {}_{uv} \\
[\bar{Y}_{rf}] & [\bar{Y}_{rr}] \\
{}_{vu} & {}_{vv}
\end{array}
\right]
\left\{
\begin{array}{c}
\{F_f - F_f^{\ b}\} \\
{}_{u1} \\
\{R\} \\
{}_{v1}
\end{array}
\right\} .
$$ (45)

The objective is to determine the unknown [Y] of the "free" system from the "measured" [Ȳ] of the restrained system and the known restraint impedance $[Z_e]$. This may be accomplished by considering the effect of the restraint system as reaction forces on the free system as shown in Figure 3c when $\{F_r'\}$ and $\{F_e'\}$ are respectively the externally applied forces on the free and the restraint system prior to connection, and $\{F_e^{\ b}\}$ and $[Z_e]\{\bar{V}_r\}$ are the reaction forces due to the restraint system. Accordingly, the net forces acting at the r coordinates of the free system of Figure 3c are

$$\{F_r\} = \{F_r' + F_e'\} - \{F_e^{\ b}\} - [Z_e]\{\bar{V}_r\} .$$ (46)

Using Equation 44,

$$\{F_r\} = \{R + F_r^{\ b}\} - [Z_e]\{\bar{V}_r\} ,$$ (47)

or

$$\{R\} = \{F_r - F_r^{\ b}\} + [Z_e]\{\bar{V}_r\} .$$ (48)

The substitution of Equation 48 in Equation 45 yields

$$
\left\{
\begin{array}{c}
\{\bar{V}_f\} \\
{}_{u1} \\
\{\bar{V}_r\} \\
{}_{v1}
\end{array}
\right\}
=
\left[
\begin{array}{cc}
[\bar{Y}_{ff}] & [\bar{Y}_{fr}] \\
{}_{uu} & {}_{uv} \\
[\bar{Y}_{rf}] & [\bar{Y}_{rr}] \\
{}_{vu} & {}_{vv}
\end{array}
\right]
\left\{
\begin{array}{c}
\{F_f - F_f^{\ b}\} \\
{}_{u1} \\
\{F_r - F_r^{\ b}\} + [Z_e]\{\bar{V}_r\} \\
{}_{v1}
\end{array}
\right\} .
$$ (49)

The determination of $[Y]$ from $[\overline{Y}]$ may be accomplished by suitably comparing Equation 49 with Equation 42. This is successfully performed by first noting that, as the result of the equivalence of Figures 3b and 3c,

$$\{\overline{V}\} \;=\; \{V\} \;,$$

and then operating on Equation 49 to yield the form of Equation 42. This operation yields the desired relation:

$$
\begin{bmatrix} [Y_{ff}]_{uu} & [Y_{fr}]_{uv} \\[2mm] [Y_{rf}]_{vu} & [Y_{rr}]_{vv} \end{bmatrix}
=
\begin{bmatrix} [I]_{uu} & [\overline{Y}_{fr}]_{uv}\,[[Y_e]_{vv} - [\overline{Y}_{rr}]]^{-1} \\[2mm] [0]_{vu} & [Y_e]_{vv}\,[[Y_e]_{vv} - [\overline{Y}_{rr}]]^{-1} \end{bmatrix}
\begin{bmatrix} [\overline{Y}_{ff}]_{uu} & [\overline{Y}_{fr}]_{uv} \\[2mm] [\overline{Y}_{rf}]_{vu} & [\overline{Y}_{rr}]_{vv} \end{bmatrix}
. \qquad (50)
$$

Frequently it is desirable to determine the effects on the system's mobility matrix as a result of some known partial restraints. This is accomplished by solving $[\overline{Y}]$ in terms of $[Y]$ in a manner similar to the above. Accordingly the result is:

$$
\begin{bmatrix} [\overline{Y}_{ff}]_{uu} & [\overline{Y}_{fr}]_{uv} \\[2mm] [\overline{Y}_{rf}]_{vu} & [\overline{Y}_{rr}]_{vv} \end{bmatrix}
=
\begin{bmatrix} [I]_{uu} & -\,[Y_{fr}]_{uv}\,[[Y_e]_{vv} + [Y_{rr}]]^{-1} \\[2mm] [0]_{vu} & [Y_e]_{vv}\,[[Y_e]_{vv} + [Y_{rr}]]^{-1} \end{bmatrix}
\begin{bmatrix} [Y_{ff}]_{uu} & [Y_{fr}]_{uv} \\[2mm] [Y_{rf}]_{vu} & [Y_{rr}]_{vv} \end{bmatrix}
. \qquad (51)
$$

If all coordinates are restrained, Equation 51 reduces to

$$[\overline{Y}_{rr}]^{-1} \;=\; [Y_{rr}]^{-1} + [Y_e]^{-1} \;,$$

a form of the Interconnection Matrix previously defined by Equation 26. The matrix of impedance $[Z]$ or $[\overline{Z}]$ may be obtained by application of the matrix inverse relationship between the impedance and mobility matrix, i.e.

$$
\begin{bmatrix} [Z_{ff}] & [Z_{fr}] \\[2mm] [Z_{rf}] & [Z_{rr}] \end{bmatrix}
\begin{bmatrix} [Y_{ff}] & [Y_{fr}] \\[2mm] [Y_{rf}] & [Y_{rr}] \end{bmatrix}
= [I] \;,
$$

or

$$
\begin{bmatrix} [\overline{Z}_{ff}] & [\overline{Z}_{fr}] \\[2mm] [\overline{Z}_{rf}] & [\overline{Z}_{rr}] \end{bmatrix}
\begin{bmatrix} [\overline{Y}_{ff}] & [\overline{Y}_{fr}] \\[2mm] [\overline{Y}_{rf}] & [\overline{Y}_{rr}] \end{bmatrix}
= [I] \;.
$$

17

9. DETERMINATION OF FREE VELOCITY
IN PRESENCE OF EXTERNAL RESTRAINTS

Frequently the free velocity matrix $\{V_{E0}\}$ of an active system cannot be determined readily under ideal free conditions. It is therefore desirable to develop a formal relationship to determine the free velocity matrix of the system from the velocity matrix determined with the system restrained in a known fashion.

As in the previous section, the desired relationship may be obtained by considering some of the exterior coordinates f to be free and the remaining coordinates r to be restrained in a known fashion (Figure 3). Thus, when the system is free, Equation 16 describing the free velocity of the active system may be written as

$$
\begin{Bmatrix} \{V_f\} - \{V_{0,f}\} \\ {}_{u1} \quad\quad {}_{u1} \\ \{V_r\} - \{V_{0,r}\} \\ {}_{v1} \quad\quad {}_{v1} \end{Bmatrix}
=
\begin{bmatrix} [Y_{ff}] & [Y_{fr}] \\ {}_{uu} & {}_{uv} \\ [Y_{rf}] & [Y_{rr}] \\ {}_{vu} & {}_{vv} \end{bmatrix}
\begin{Bmatrix} \{F_f\} \\ {}_{u1} \\ \{F_r\} \\ {}_{v1} \end{Bmatrix} ,
\tag{52}
$$

which is obtained by partitioning in terms of submatrices corresponding to free and restrained coordinates. When the system is restrained by impedance $[Z_e]$ as shown in Figure 3b, the effect of the restraint system on the free system may be considered in terms of reaction forces as shown in Figure 3c. Thus, the net forces acting on the free system at the restrained coordinates are given by Equation 46. Accordingly, replacing $\{F_r\}$ in Equation 52 by Equation 46, and denoting the measurement of velocities $\{V_f\}$ and $\{V_r\}$ under the restrained conditions as $\{\bar{V}_f\}$ and $\{\bar{V}_r\}$ respectively, yields

$$
\begin{Bmatrix} \{\bar{V}_f\} - \{V_{0,f}\} \\ \{\bar{V}_r\} - \{V_{0,r}\} \end{Bmatrix}
=
\begin{bmatrix} [Y_{ff}] & [Y_{fr}] \\ [Y_{rf}] & [Y_{rr}] \end{bmatrix}
\begin{Bmatrix} \{F_f\} \\ \{F_r' + F_e'\} - [Z_e]\,\bar{V}_r - \{F_e^b\} \end{Bmatrix} .
\tag{53}
$$

For free velocity measurements at the free and restrained coordinates,

$$
\{F_f\} = \{F_r'\} = \{F_e'\} = 0 .
\tag{54}
$$

Substituting the condition of Equation 54 in Equation 53 and simplifying yields

$$
\begin{Bmatrix} \{V_{0,f}\} \\ {}_{u1} \\ \{V_{0,r}\} \\ {}_{v1} \end{Bmatrix}
=
\begin{bmatrix} [I_{ff}] & [Y_{fr}]\,[Y_e]^{-1} \\ {}_{uu} & {}_{uv} \\ [0] & [I_{rr}] + [Y_{rr}]\,[Y_e]^{-1} \\ {}_{vu} & {}_{vv} \end{bmatrix}
\begin{Bmatrix} \{\bar{V}_f\} \\ {}_{u1} \\ \{\bar{V}_r\} \\ {}_{v1} \end{Bmatrix}
+
\begin{Bmatrix} [Y_{fr}]\,\{F_e^b\} \\ {}_{uv} \quad {}_{v1} \\ [Y_{rr}]\,\{F_e^b\} \\ {}_{vv} \quad {}_{v1} \end{Bmatrix} ,
\tag{55}
$$

which is the desired relation of free velocities in terms of velocities measured under the restraint condition. The procedure consists of first determining the mobility of the free system [Y] by the

method of Section 8 and the blocked forces of the active restraint system.* Next, measure the velocities of the active system at the free and restrained coordinates $\{\bar{V}_f\}$ and $\{\bar{V}_r\}$ respectively and apply Equation 55 to obtain the true free velocities $\{V_{0,f}\}$ and $\{V_{0,r}\}$ of the active system using the known mobility $[Y]$. It may be seen that if all coordinates are free $\left([Y_e] \to \infty\right)$,

$$\{\bar{V}_f\} \to \{V_{0,f}\} \, ,$$

and

$$\{\bar{V}_r\} \to \{V_{0,r}\} \, ,$$

as would be expected.

10. EXAMPLES OF APPLICATION

10.1. Analytical Approach

Consider the two dimensional system of Figure 4 in which two bodies M_1 and M_2 are supported by elastic and damping elements located at their corners, as illustrated. The system is symmetrical with respect to the axis passing through the center of gravity of the bodies. Vibration of the system results from an oscillating force acting upon M_1. The force is applied at a point on the plane of symmetry, at a distance ϵ above the center of gravity of M_1 and has a horizontal component $f_x \cos \omega t$, and a vertical component $f_y \sin \omega t$.

The degrees-of-freedom of the center of gravity are as shown in the upper right corner of each body, and for small distortions about the system's stable equilibrium, these are taken as generalized displacements x_β ($\beta = 1, 2, \cdots, 6$). In this example the coordinates x_4, x_5, and x_6 will be referred to as "exterior."

10.1.1 Energy Expressions

The kinetic, potential and dissipation energy of the system are given, respectively,
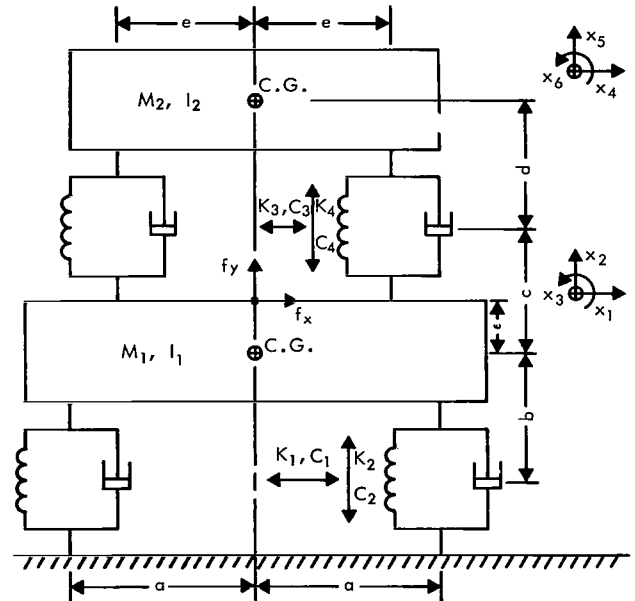

Figure 4—Lumped parameter equivalent of a simplified active two-dimensional system.

---

*Note that $\{F_e^b\}$ can be obtained from free velocity measurements using Equation 13.

19

by:

$$2T = M_1 \left(\dot{x}_1^2 + \dot{x}_2^2 \right) + I_1 \dot{x}_3^2 + M_2 \left(\dot{x}_4^2 + \dot{x}_5^2 \right) + I_2 \dot{x}_6^2 \ ,$$

$$2P = 2K_1 x_1^2 + 2K_1 b^2 x_3^2 + 2K_2 x_2^2 + 2K_2 a^2 x_3^2$$

$$+ 2K_3 \left(x_1 - x_4 \right)^2 + 2K_3 \left(cx_3 - dx_6 \right)^2$$

$$+ 2K_4 \left(x_2 - x_5 \right)^2 + 2K_4 \left(ex_3 - ex_6 \right)^2 \ ,$$

$$2D = 2C_1 \dot{x}_1^2 + 2C_1 b^2 \dot{x}_3^2 + 2C_2 \dot{x}_2^2 + 2C_2 a^2 \dot{x}_3^2$$

$$+ 2C_3 \left(\dot{x}_1 - \dot{x}_4 \right)^2 + 2C_3 \left(c\dot{x}_3 - d\dot{x}_6 \right)^2$$

$$+ 2C_4 \left(\dot{x}_2 - \dot{x}_5 \right)^2 + 2C_4 \left(e\dot{x}_3 - e\dot{x}_6 \right)^2 \ . \tag{56}$$

10.1.2 Equations of Motion

Applying Lagrange's equation for nonconservative systems yields the equations of motion:

$$M_1 \ddot{x}_1 + 2\left(C_1 + C_3 \right) \dot{x}_1 - 2C_3 \dot{x}_4 + 2\left(K_1 + K_3 \right) x_1 - 2K_3 x_4 = f_1 \ ,$$

$$M_1 \ddot{x}_2 + 2\left(C_2 + C_4 \right) \dot{x}_2 - 2C_4 \dot{x}_5 + 2\left(K_2 + K_4 \right) x_2 - 2K_4 x_5 = f_2 \ ,$$

$$I_1 \ddot{x}_3 + 2\left(C_1 b^2 + C_2 a^2 + C_3 c^2 + C_4 e^2 \right) \dot{x}_3 - 2\left(C_3 cd + C_4 e^2 \right) \dot{x}_6$$

$$+ 2\left(K_1 b^2 + K_2 a^2 + K_3 c^2 + K_4 e^2 \right) x_3 - 2\left(K_3 cd + K_4 e^2 \right) x_6 = f_3 \ ,$$

$$M_2 \ddot{x}_4 - 2C_3 \dot{x}_1 + 2C_3 \dot{x}_4 - 2K_3 x_1 + 2K_3 x_4 = f_4 \ ,$$

$$M_2 \ddot{x}_5 - 2C_4 \dot{x}_2 + 2C_4 \dot{x}_5 - 2K_4 x_2 + 2K_4 x_5 = f_5 \ ,$$

$$I_2 \ddot{x}_6 - 2\left(C_3 cd + C_4 e^2 \right) \dot{x}_3 + 2\left(C_3 d^2 + C_4 e^2 \right) \dot{x}_6$$

$$- 2\left(K_3 cd + K_4 e^2 \right) x_3 + 2\left(K_3 d^2 + K_4 e^2 \right) x_6 = f_6 \ , \tag{57}$$

where f_α ($\alpha = 1, 2, \cdots, 6$) are generalized forces to be determined later.

10.1.3 Coefficient Matrices

From the equations of motion, the coefficient matrices are obtained as:

$[m_{\alpha\beta}]$:

$$m_{11} = m_{22} = M_1 \ ,$$

$$m_{33} = I_1 \ ,$$

$$m_{44} = m_{55} = M_2 \ ,$$

$$m_{66} = I_2 \ ,$$

$$m_{\alpha\beta} = 0 \ ; \quad \alpha \neq \beta(\alpha, \ \beta = 1, \ 2, \ \cdots, 6) \ . \tag{58}$$

$[k_{\alpha\beta}]$:

$$k_{11} = 2(K_1 + K_3) \ ,$$

$$k_{22} = 2(K_2 + K_4) \ ,$$

$$k_{33} = 2(K_1 \, b^2 + K_2 \, a^2 + K_3 \, c^2 + k_4 \, e^2) \ ,$$

$$k_{44} = 2K_3 \ ,$$

$$k_{55} = 2K_4 \ ,$$

$$k_{66} = 2(K_3 \, d^2 + K_4 \, e^2) \ ,$$

$$k_{14} = k_{41} = - 2K_3 \ ,$$

$$k_{25} = k_{52} = - 2K_4 \ ,$$

$$k_{36} = k_{63} = - 2(K_3 \, cd + K_4 \, e^2) \ .$$

All others are zero. (59)

21

$$[c_{\alpha\beta}]:$$

$$c_{11} = 2(C_1 + C_3) \ ,$$

$$c_{22} = 2(C_2 + C_4) \ ,$$

$$c_{33} = 2(C_1 b^2 + C_2 a^2 + C_3 c^2 + C_4 e^2) \ ,$$

$$c_{44} = 2C_3 \ ,$$

$$c_{55} = 2C_4 \ ,$$

$$c_{66} = 2(C_3 d^2 + C_4 e^2) \ ,$$

$$c_{14} = c_{41} = -2C_3 \ ,$$

$$c_{25} = c_{52} = -2C_4 \ ,$$

$$c_{36} = c_{63} = -2(C_3 cd + C_4 e^2) \ .$$

All others are zero. (60)

From Equation 4,

$$[Z_{\alpha\beta}]:$$

$$Z_{11} = i\omega M_1 + 2(C_1 + C_3) + 2(K_1 + K_3)/i\omega \ , \qquad (Z_{II}^{\delta\delta})$$

$$Z_{22} = i\omega M_1 + 2(C_2 + C_4) + 2(K_2 + K_4)/i\omega \ , \qquad (Z_{II}^{\delta\delta})$$

$$Z_{33} = i\omega I_1 + 2(C_1 b^2 + C_2 a^2 + C_3 c^2 + C_4 e^2) + 2(K_1 b^2 + K_2 a^2 + K_3 c^2 + K_3 e^2)/i\omega \ , \qquad (Z_{II}^{\theta\theta})$$

$$Z_{44} = i\omega M_2 + 2C_3 + 2K_3/i\omega \ , \qquad (Z_{EE}^{\delta\delta})$$

$$Z_{55} = i\omega M_2 + 2C_4 + 2K_4/i\omega \ , \qquad (Z_{EE}^{\delta\delta})$$

$$Z_{66} = i\omega I_2 + 2(C_3 d^2 + C_4 e^2) + 2(K_3 d^2 + K_4 e^2)/i\omega \ , \qquad (Z_{EE}^{\theta\theta})$$

$$Z_{14} = Z_{41} = -2(C_3 + K_3/i\omega) \ , \qquad (Z_{IE}^{\delta\delta}, Z_{EI}^{\delta\delta})$$

$$Z_{25} = Z_{52} = -2(C_4 + K_4/i\omega) \ , \qquad (Z_{IE}^{\delta\delta}, Z_{EI}^{\delta\delta})$$

$$Z_{36} = Z_{63} = -2(C_3 cd + C_4 e^2) - 2(K_3 cd + K_4 e^2)/i\omega \ . \qquad (Z_{IE}^{\theta\theta}, Z_{EI}^{\theta\theta})(61)$$

The type of impedance has been designated on the extreme right. Accordingly, rearranging rows and columns, and subsequent partitioning of $\left[Z_{\alpha\beta}\right]$ yields the following submatrices:*

$$\left[Z_{EE}\right] = \begin{bmatrix} Z_{44} & 0 & 0 \\ 0 & Z_{55} & 0 \\ 0 & 0 & Z_{66} \end{bmatrix},$$

$$\left[Z_{EI}\right] = \begin{bmatrix} Z_{41} & 0 & 0 \\ 0 & Z_{52} & 0 \\ 0 & 0 & Z_{63} \end{bmatrix},$$

$$\left[Z_{IE}\right] = \begin{bmatrix} Z_{14} & 0 & 0 \\ 0 & Z_{25} & 0 \\ 0 & 0 & Z_{36} \end{bmatrix},$$

$$\left[Z_{II}\right] = \begin{bmatrix} Z_{11} & 0 & 0 \\ 0 & Z_{22} & 0 \\ 0 & 0 & Z_{33} \end{bmatrix},$$

$$\left\{V_E\right\} = \begin{Bmatrix} V_4 \\ V_5 \\ V_6 \end{Bmatrix}, \qquad \left\{V_I\right\} = \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \end{Bmatrix},$$

$$\left\{F_E\right\} = \begin{Bmatrix} F_4 \\ F_5 \\ F_6 \end{Bmatrix}, \qquad \left\{F_I\right\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}. \tag{62}$$

10.1.4 Generalized Forces

The total virtual work done by the applied forces on the system is

$$\delta W_T = f_y\,\delta x_2 + f_x\,\delta x_1 - f_x\,\epsilon\,\delta x_3 . \tag{63}$$

---

*These submatrices are required by Equation 10 to determine $\left[Z_E\right]$.

The generalized force f_α (corresponding to x_α) is defined by

$$\delta W_\alpha = f_\alpha \, \delta x_\alpha \, , \tag{64}$$

from which $f_1 = f_x$, $f_2 = f_y$, $f_3 = -f_x \epsilon$. Thus

$$\{F_I\} = \left\{ \begin{array}{c} F_x \\ F_y \\ -\epsilon F_x \end{array} \right\} \, , \qquad \{F_E\} = \{0\} \, , \tag{65}$$

where F_x, F_y and $-\epsilon F_x$ are the Fourier transforms of f_1, f_2 and f_3 respectively.

10.1.5 Determination of $[Z_{II}]^{-1}$

In Equation 10, the inverse matrix of $[Z_{II}]$ is required for the determination of $[Z_E]$.* Let us denote the elements of $[Z_{II}]^{-1}$ by $h_{\alpha\beta}$. Then $h_{\alpha\beta}$ is related to $Z_{\alpha\beta}$ by

$$\sum_{k=1}^{3} Z_{II,\alpha k} \, h_{k\beta} = \delta_{\alpha\beta} \, , \tag{66}$$

where $\delta_{\alpha\beta}$ is the so-called "Kronecker delta."

Since $[Z_{II}]$ is a diagonal matrix, Equation 66 reduces to

$$Z_{II,\alpha\alpha} h_{\alpha\alpha} = 1 \, .$$

Thus

$$[h_{\alpha\beta}] = \begin{bmatrix} \dfrac{1}{Z_{11}} & 0 & 0 \\[2mm] 0 & \dfrac{1}{Z_{22}} & 0 \\[2mm] 0 & 0 & \dfrac{1}{Z_{33}} \end{bmatrix} \, . \tag{67}$$

10.1.6 Determination of $[Z_E]$

From Equation 10, the general element of $[Z_E]$ is given by

$$Z_{E,\alpha\beta} = Z_{EE,\alpha\beta} - \sum_{\ell=1}^{3} \sum_{k=1}^{3} Z_{EI,\alpha k} \, h_{II,k\ell} \, Z_{IE,\ell\beta} \, . \qquad (\alpha, \beta = 1, 2, 3) \tag{68}$$

---

*The inversion for this particular example is easily obtained from Equation 66. In general, inversion of higher orders may require machine computation.

Since, for this example:

$$Z_{EE,\alpha\beta} = 0 \qquad (\alpha \neq \beta) ,$$

$$Z_{EI,\alpha k} = 0 \qquad (\alpha \neq k) ,$$

$$h_{II,k\ell} = 0 \qquad (k \neq \ell) ,$$

$$Z_{IE,\ell\beta} = 0 \qquad (\ell \neq \beta) ,$$

then

$$Z_{E,\alpha\beta} = 0 \qquad (\alpha \neq \beta) ,$$

and

$$Z_{E,\alpha\alpha} = Z_{EE,\alpha\alpha} - Z_{EI,\alpha\alpha}^2 h_{II,\alpha\alpha} \qquad (\alpha = \beta = 1,\, 2,\, 3) .$$

Thus

$$\left[Z_{E,\alpha\beta} \right] = \begin{bmatrix} Z_{44} - Z_{14}^2 / Z_{11} & 0 & 0 \\ 0 & Z_{55} - Z_{25}^2 / Z_{22} & 0 \\ 0 & 0 & Z_{66} - Z_{36}^2 / Z_{33} \end{bmatrix} . \qquad (69)$$

10.1.7 Determination of $\left\{ F_{E,\alpha}^b \right\}$

From Equation 11,

$$F_{E,\alpha}^b = \sum_{\ell=1}^{3} \sum_{k=1}^{3} Z_{EI,\alpha k} h_{II,k\ell} F_{I,\ell} \qquad (\alpha = 1,\, 2,\, 3) , \qquad (70)$$

but for this example, $F_{E,\alpha}^b = Z_{EI,\alpha\alpha} h_{II,\alpha\alpha} F_{I,\alpha}$, from which

$$\left\{ F_{E,\alpha}^b \right\} = \left\{ \begin{array}{c} \left(Z_{14} / Z_{11} \right) F_x \\ \left(Z_{25} / Z_{22} \right) F_y \\ - \left(Z_{36} / Z_{33} \right) \epsilon F_x \end{array} \right\} . \qquad (71)$$
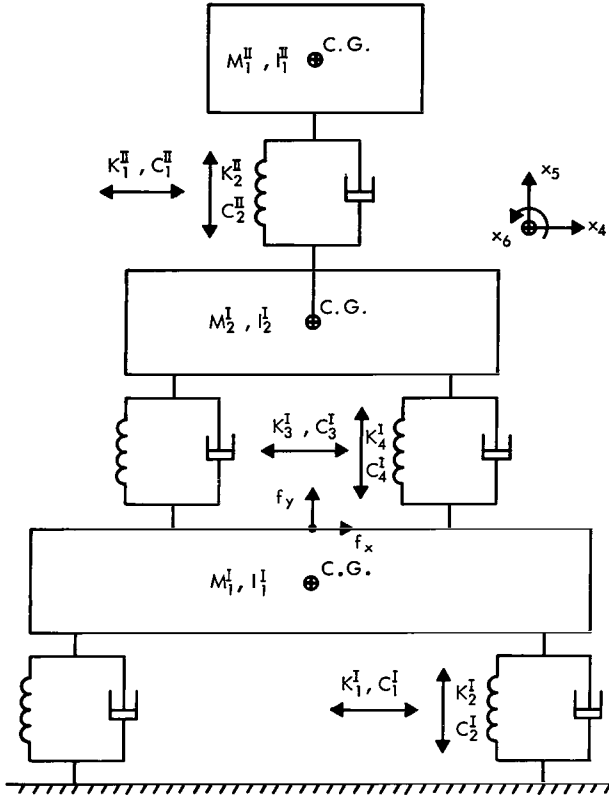
Figure 5—Lumped parameter equivalent of the composite system composed of systems I and II, with connection at center of gravity of M_2^I.

Figure 6—Lumped parameter equivalent of a simplified passive two-dimensional system.

10.1.8 Determination of $\{V_E\}$

From Equation 12,

$$\{V_{E,\alpha}\} = \begin{Bmatrix} V_4 \\ V_5 \\ V_6 \end{Bmatrix} = \begin{Bmatrix} -Z_{14}\left(Z_{11}\,Z_{44}-Z_{14}^2\right)^{-1}F_x \\ -Z_{25}\left(Z_{22}\,Z_{55}-Z_{25}^2\right)^{-1}F_y \\ Z_{36}\left(Z_{33}\,Z_{66}-Z_{36}^2\right)^{-1}\epsilon\,F_x \end{Bmatrix} . \quad (72)$$

10.1.9 Effect of Subsystems

Next we shall consider the effect of connecting a second system to the exterior coordinates x_4, x_5, and x_6 (Figure 5). The system is assumed to be passive (Figure 6).

Letting superscripts I and II denote quantities corresponding respectively to the first and second system, the following is obtained:

$$\left[Z_{E,\alpha\beta}^{II}\right] = \begin{bmatrix} Z_{44}^{II}-\left(Z_{14}^{II}\right)^2\big/Z_{11}^{II} & 0 & 0 \\ 0 & Z_{55}^{II}-\left(Z_{25}^{II}\right)^2\big/Z_{22}^{II} & 0 \\ 0 & 0 & Z_{66}^{II}-\left(Z_{36}^{II}\right)^2\big/Z_{33}^{II} \end{bmatrix},$$

where

$$Z_{11}^{II} = i\omega M_1^{II} + C_1^{II} + K_1^{II}\big/i\omega ,$$
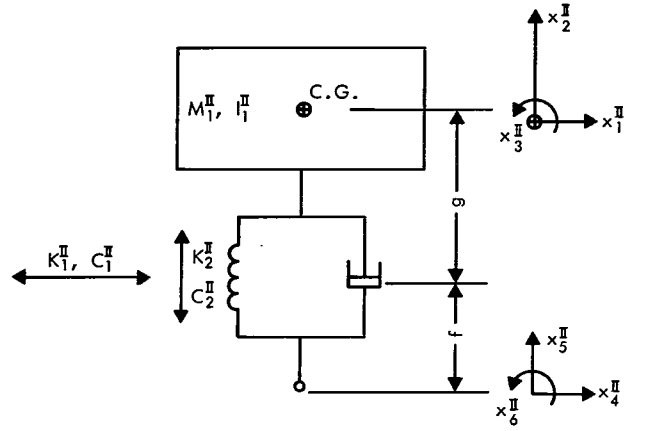
$$Z_{22}^{II} = i\omega M_1^{II} + C_2^{II} + K_2^{II}\big/i\omega ,$$

26

$$Z_{33}^{II} = i\omega I_1^{II} + g^2 \left(C_1^{II} + K_1^{II} / i\omega \right) ,$$

$$Z_{44}^{II} = C_1^{II} + K_1^{II} / i\omega ,$$

$$Z_{55}^{II} = C_2^{II} + K_2^{II} / i\omega ,$$

$$Z_{66}^{II} = f^2 \left(C_1^{II} + K_1^{II} / i\omega \right) ,$$

$$Z_{14}^{II} = - \left(C_1^{II} + K_1^{II} / i\omega \right) ,$$

$$Z_{25}^{II} = - \left(C_2^{II} + K_2^{II} / i\omega \right) ,$$

$$Z_{36}^{II} = - gf \left(C_1^{II} + K_1^{II} / i\omega \right) . \tag{73}$$

From Equation 26, the interconnection matrix is

$$\left[H_{\alpha\beta} \right] = \begin{bmatrix} \displaystyle\sum_{L=I}^{II} \left[Z_{44}^{L} - \left(Z_{14}^{L} \right)^2 / Z_{11}^{L} \right] & 0 & 0 \\[20pt] 0 & \displaystyle\sum_{L=I}^{II} \left[Z_{55}^{L} - \left(Z_{25}^{L} \right)^2 / Z_{22}^{L} \right] & 0 \\[20pt] 0 & 0 & \displaystyle\sum_{L=I}^{II} \left[Z_{66}^{L} - \left(Z_{36}^{L} \right)^2 / Z_{33}^{L} \right] \end{bmatrix} . \tag{74}$$

The roots of $\Delta(H) = 0$ are the critical frequencies of the composite system.

If $Q_{\alpha\beta}$ denotes the elements of $\left[H_{\alpha\beta} \right]^{-1}$, then

$$\left[Q_{\alpha\beta} \right] = \begin{bmatrix} 1/H_{11} & 0 & 0 \\ 0 & 1/H_{22} & 0 \\ 0 & 0 & 1/H_{33} \end{bmatrix} . \tag{75}$$

It follows from Equation 23 and 25 that the response of the composite system at the interconnection of the two systems is:

$$V_4 = -\left(Z_{14}^{I}/Z_{11}^{I}\right) F_x \bigg/ \sum_{L=I}^{II}\left[Z_{44}^{L} - \left(Z_{14}^{L}\right)^2/Z_{11}^{L}\right] ,$$

$$V_5 = -\left(Z_{25}^{I}/Z_{22}^{I}\right) F_y \bigg/ \sum_{L=I}^{II}\left[Z_{55}^{L} - \left(Z_{25}^{L}\right)^2/Z_{32}^{L}\right] ,$$

$$V_6 = \left(Z_{36}^{I}/Z_{33}^{I}\right) \epsilon F_x \bigg/ \sum_{L=I}^{II}\left[Z_{66}^{L} - \left(Z_{36}^{L}\right)^2/Z_{33}^{L}\right] . \tag{76}$$

10.2. Measurement Approach

The primary purpose here will be to demonstrate the experimental evaluation of the mobility matrix $[Y_E]$ of each of the systems, to obtain their corresponding interconnection matrix and to describe the resulting response at the interconnection.

10.2.1 Determination of Mobility Parameters

Equation 35 for passive systems* is

$$\left\{V^{(k)}\left(\omega_0\right)\right\} = \left[Y\left(\omega_0\right)\right]\left\{F^{(k)}\left(\omega_0\right)\right\} , \tag{35}$$

where $k = 3$ since there are three exterior coordinates in each system.

10.2.2 System I

With the system suspended on a low-frequency suspension such that all exterior coordinates (x_4, x_5, x_6) are unattached (zero load impedance or infinite mobility), apply a sinusoidal force of constant frequency ω_0 at x_4 and measure the phasor (referencing phase angle with respect to the applied sinusoidal force) of the resulting velocity at x_4, x_5 and x_6. Denote these velocity phasors and this single applied force phasor respectively by

$$V_4^{(1)}\left(\omega_0\right) = \left|V_4^{(1)}\left(\omega_0\right)\right| \angle \theta_{V_4}^{(1)}\left(\omega_0\right) ,$$

$$V_5^{(1)}\left(\omega_0\right) = \left|V_5^{(1)}\left(\omega_0\right)\right| \angle \theta_{V_5}^{(1)}\left(\omega_0\right) ,$$

$$V_6^{(1)}\left(\omega_0\right) = \left|V_6^{(1)}\left(\omega_0\right)\right| \angle \theta_{V_6}^{(1)}\left(\omega_0\right) ,$$

$$F_4^{(1)}\left(\omega_0\right) = \left|F_4^{(1)}\left(\omega_0\right)\right| \angle \theta_{F_4}^{(1)}\left(\omega_0\right) = 0 . \tag{77}$$

---

*Although system I is truly active, the use of Equation 35 for passive systems supplemented by an independent determination of free velocities is equally applicable.

28

Repeating the above test two more times, each time applying the sinusoidal force of frequency ω_0 successively at x_5 and x_6, provides the additional set of measurements similar to (77):

$$V_4^{(2)}\left(\omega_0\right) = \left|V_4^{(2)}\left(\omega_0\right)\right| \underline{\bigg/\theta_{V_4^{(2)}}\left(\omega_0\right)} \ ,$$

$$V_5^{(2)}\left(\omega_0\right) = \left|V_5^{(2)}\left(\omega_0\right)\right| \underline{\bigg/\theta_{V_5^{(2)}}\left(\omega_0\right)} \ ,$$

$$V_6^{(2)}\left(\omega_0\right) = \left|V_6^{(2)}\left(\omega_0\right)\right| \underline{\bigg/\theta_{V_6^{(2)}}\left(\omega_0\right)} \ ,$$

$$F_5^{(2)}\left(\omega_0\right) = \left|F_5^{(2)}\left(\omega_0\right)\right| \underline{\bigg/\theta_{F_5^{(2)}}\left(\omega_0\right)} = 0 \ , \tag{78}$$

$$V_4^{(3)}\left(\omega_0\right) = \left|V_4^{(3)}\left(\omega_0\right)\right| \underline{\bigg/\theta_{V_4^{(3)}}\left(\omega_0\right)} \ ,$$

$$V_5^{(3)}\left(\omega_0\right) = \left|V_5^{(3)}\left(\omega_0\right)\right| \underline{\bigg/\theta_{V_5^{(3)}}\left(\omega_0\right)} \ ,$$

$$V_6^{(3)}\left(\omega_0\right) = \left|V_6^{(3)}\left(\omega_0\right)\right| \underline{\bigg/\theta_{V_6^{(3)}}\left(\omega_0\right)} \ ,$$

$$F_6^{(3)}\left(\omega_0\right) = \left|F_6^{(3)}\left(\omega_0\right)\right| \underline{\bigg/\theta_{F_6^{(3)}}\left(\omega_0\right)} = 0 \ . \tag{79}$$

From (77), (78) and (79) the elements of the mobility matrix are simply computed as

$$Y_{44}\left(\omega_0\right) = \frac{\left|V_4^{(1)}\left(\omega_0\right)\right|}{\left|F_4^{(1)}\left(\omega_0\right)\right|} \underline{\bigg/\theta_{V_4^{(1)}}\left(\omega_0\right)} \ ,$$

$$Y_{54}\left(\omega_0\right) = \frac{\left|V_5^{(1)}\left(\omega_0\right)\right|}{\left|F_4^{(1)}\left(\omega_0\right)\right|} \underline{\bigg/\theta_{V_5^{(1)}}\left(\omega_0\right)} \ ,$$

$$Y_{64}\left(\omega_0\right) = \frac{\left|V_6^{(1)}\left(\omega_0\right)\right|}{\left|F_4^{(1)}\left(\omega_0\right)\right|} \underline{\bigg/\theta_{V_6^{(1)}}\left(\omega_0\right)} \ ,$$

$$Y_{45}\left(\omega_0\right) = \frac{\left|V_4^{(2)}\left(\omega_0\right)\right|}{\left|F_5^{(2)}\left(\omega_0\right)\right|} \underline{\bigg/\theta_{V_4^{(2)}}\left(\omega_0\right)} \ ,$$

$$Y_{55}\left(\omega_0\right) = \frac{\left|V_5^{(2)}\left(\omega_0\right)\right|}{\left|F_5^{(2)}\left(\omega_0\right)\right|} \underline{\bigg/\theta_{V_5^{(2)}}\left(\omega_0\right)} \ ,$$

$$Y_{65}(\omega_0) = \frac{\left|V_6^{(2)}(\omega_0)\right|}{\left|F_5^{(2)}(\omega_0)\right|} \underline{/\theta_{V_6^{(2)}}(\omega_0)} \ ,$$

$$Y_{46}(\omega_0) = \frac{\left|V_4^{(3)}(\omega_0)\right|}{\left|F_6^{(3)}(\omega_0)\right|} \underline{/\theta_{V_4^{(3)}}(\omega_0)} \ ,$$

$$Y_{56}(\omega_0) = \frac{\left|V_5^{(3)}(\omega_0)\right|}{\left|F_6^{(3)}(\omega_0)\right|} \underline{/\theta_{V_5^{(3)}}(\omega_0)} \ ,$$

$$Y_{66}(\omega_0) = \frac{\left|V_6^{(3)}(\omega_0)\right|}{\left|F_6^{(3)}(\omega_0)\right|} \underline{/\theta_{V_6^{(3)}}(\omega_0)} \ . \tag{80}$$

In Equation 80 the condition $Y_{\alpha\beta} = Y_{\beta\alpha}$ $(\alpha \neq \beta)$ provides a check on the linearity of the system.

The complete determination of the mobility matrix requires repeating the above procedures for all frequencies of interest.

10.2.3 System II

The determination of the mobility matrix for system II is accomplished in a manner similar to that for system I. The exterior coordinates of this system are defined as x_4^{II}, x_5^{II} and x_6^{II} as shown in Figure 6.

10.2.4 Determination of Free Velocities

In this example, it is necessary only to determine the free velocities of system I since system II is passive. With the system suspended as before, the resulting free velocity phasor at all exterior coordinates due to the excitations f_x and f_y at frequency ω_0 may be measured directly. A convenient reference for phase angle measurements is the free velocity at one of the coordinates. Assuming that the phase angle of these measurements is made relative to the free velocity at coordinate x_4, the free velocity phasors may be expressed as

$$V_{0,4}(\omega_0) = \left|V_{0,4}(\omega_0)\right| \underline{/\theta_{v0,4}(\omega_0)} = 0 \ ,$$

$$V_{0,5}(\omega_0) = \left|V_{0,5}(\omega_0)\right| \underline{/\theta_{v0,5}(\omega_0)} \ ,$$

$$V_{0,6}(\omega_0) = \left|V_{0,6}(\omega_0)\right| \underline{/\theta_{v0,6}(\omega_0)} \ . \tag{81}$$

The complete determination of free velocity requires operating the system over all frequencies of interest and performing the above procedure.

10.2.5 Interconnection Matrix

If we denote the $[Y_E]$ for the active system and passive system respectively by superscripts I and II, the interconnection matrix may be obtained from Equation 26 as

$$[H(\omega)] = [Y^I(\omega)]^{-1} + [Y^{II}(\omega)]^{-1} , \tag{82}$$

where the matrix inversion of $[Y(\omega)]$ may require digital machine computations.

10.2.6 Prediction of Composite Response

The response at the interconnection of the two systems may be predicted by

$$\{V(\omega)\} = -[H(\omega)]^{-1}\{F^{b^I}(\omega)\} , \tag{83}$$

where $\{F^{b^I}\}$ can be obtained from

$$\{F^{b^I}(\omega)\} = [Y^I(\omega)]^{-1}\{V_0(\omega)\} . \tag{84}$$

11. DISCUSSION

The objective of this paper is to present a systematic approach for the analytic and experimental determination of mechanical impedance and its associated effects on dynamic response of complex mechanical systems. The nature of the formulation of the approach is sufficiently general for extension to cases of many types of distributed systems subject to varied steady state, transient and random excitations.

Although the concepts related to the use of mechanical impedance expressions have been explored extensively in the development of methods for shock and vibration control, they have been limited to a large extent to systems which permit analysis of only one-dimensional types (Reference 2). Limited and specialized application to multi-dimensional systems has been studied only recently. As shown in this study, the treatment of mechanical impedance of multi-degree of freedom/ multi-dimensional systems may be approached by adopting certain aspects of matrix methods of structural analysis, together with appropriate interpretation of quantities related to the concepts of mechanical impedance (e.g., blocked force, free velocity, point and transfer impedance).

The problem of describing complex mechanical systems so as to consider the information available about the individual subsystems, involves the formulation of impedance matrices and interconnection equations wherein the conditions of force equilibrium and velocity compatibility, implicitly imposed, are equivalent to physical connection of the subsystems. The characteristic values of the interconnection matrix correspond to critical frequencies of the composite systems as observed at the interconnection. Moreover, the methods of interconnection developed are applicable to many systems of complex combinations. Systems of mixed types such as electro-mechanical

and acoustic-mechanical may be treated using these methods. With regard to environmental testing, the methods of interconnection further provide a sound theoretical basis for performing subassembly testing which would obviate the demand for increasingly large complex environmental test facilities for future generation aerospace systems.

On the basis of the results of this study, the following recommendations are made:

1. Suitable experiments should be undertaken to determine the proper extent of the practical application of the methods herein presented.

2. Effort should be directed toward developing a computer program for mathematical computation of the required matrix relationships.

3. Effort should be directed toward the development of sensors and/or data reduction methods for the measurement of multi-dimensional impedance parameters.

4. Effort should be directed toward the investigation of experimental techniques for determining these parameters using random and transient types of force excitations.

5. The feasibility of applying the results of this study to environmental test simulation should be investigated in further studies. Included should be a laboratory study of simulation and control of impedance of multi-dimensional models.

REFERENCES

1. On, F. J. and Belsheim, R. O., "A Theoretical Basis for Mechanical Impedance Simulation in Shock and Vibration Testing of One-Dimensional Systems," NASA TN D-1854, August, 1963.

2. Plunkett, R., "Colloquium on Mechanical Impedance Methods for Mechanical Vibrations," American Society of Mechanical Engineers, New York, 1958.

3. On, F. J., "Preliminary Study of An Experimental Method in Multi-dimensional Mechanical-Impedance Determination," 34th Shock and Vibration Bulletin, Naval Research Laboratory, Bulletin 34, Part 3, pp. 27-35, December, 1964, AD 460 001.

4. O'Hara, G. J., "Mechanical Impedance and Mobility Concepts," U. S. Naval Research Laboratory, Report 6406, July 29, 1966, AD 638 206.

Appendix

Symbols

a, b, c, d, e, f, g   linear dimensions for example problem

C damping coefficient

D dissipation energy

ϵ distance above c.g. of mass M_1

δ superscript for rectilinear quantities

$\delta\delta$ superscript for rectilinear couplings

$\delta\theta,\ \theta\delta$ superscripts for rectilinear—rotational couplings, or conversely

E subscript for exterior quantities

EE subscript for exterior couplings

$EI,\ IE$ subscripts for exterior-interior couplings, or conversely

f_α generalized force corresponding to αth generalized coordinate

f_x applied force in x-direction

f_y applied force in y-direction

I subscript for interior quantities

II subscript for interior couplings

I mass moment of inertia

K stiffness coefficient

M mass

N number of degrees-of-freedom

P potential energy

$F_\alpha(\omega)$ Fourier transform of $f_\alpha(t)$

$V_\beta(\omega)$ Fourier transform of $\dot{x}_\beta(t)$

n   number of exterior coordinates

θ subscript for rotational quantities

$\theta\theta$ subscript for rotational couplings

T   kinetic energy

t   time variable

I, II   superscripts for specific subsystems

ω circular frequency

x_β displacement corresponding to βth coordinate

$\dot{x}_\beta$ velocity

$\ddot{x}_\beta$ acceleration

$Z_{\alpha\beta}(\omega)$ frequency dependent characteristic property of system

δW_T total virtual work done by applied force

δx_α virtual displacement

$\{\ \}$ denotes column matrix

$\{\ \}^T$ denotes transpose of $\{\ \}$

$[\ ]$ denotes square or rectangular matrix

$[\ ]^T$ denotes transpose of $[\ ]$

$[\ ]^{-1}$ denotes inverse matrix of square matrix

$[m_{\alpha\beta}]$ inertia matrix

$[c_{\alpha\beta}]$ damping matrix

$[k_{\alpha\beta}]$ stiffness matrix

$\{F_\alpha\}$ transform force matrix

$\{F_E^b\}$ blocked force matrix (transform)

$\{V_\beta\}$ transform velocity matrix

$\{V_{E0}\}$ transform free velocity matrix

$\left[Z_{\alpha\beta}\right]$ mechanical impedance matrix for system

$\left[Z_E\right]$ impedance matrix looking back from exterior coordinates

$[Y]$ inverse matrix of $[Z]$

$\left[Z_e\right]$ impedance matrix of restraint system

$\left[H_{\alpha\beta}\right]$ interconnection matrix

$\triangle(H)$ determinant of $\left[H_{\alpha\beta}\right]$

$\left[Q_{\alpha\beta}\right]$ inverse matrix of $\left[H_{\alpha\beta}\right]$

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546